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LETTER TO THE EDITOR

On the correlation function of the 2D antiferromagnetic Potts model

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Abstract. The low-temperature ordering problem for the antiferromagnetic (AF) Potts models has been approached by computing the corner–corner correlation functions of the 3- and 4-states AF Potts models laid on finite-size square and triangular lattices respectively. The transfer matrix computations, decimating one spin at a time, have been used. The results obtained indicate that for both models spin correlations decay algebraically with distance at $T=0$.

We consider the long-range order existence problem for the 3-states antiferromagnetic (AF) Potts model [1, 2] laid on a square lattice, and for the 4-states model, laid on a triangular lattice, at $T=0$. In these models the spin variables s_i may take q values $s_i = 1, \ldots, q$, the Hamiltonian being

$$H = \tfrac{1}{2} \sum_{\langle ij \rangle} \delta(s_i, s_j) \tag{1}$$

where the summation is restricted to nearest neighbours only.

The low-temperature ordering problem has not yet been solved unambiguously for the AF models [3–8]. In both cases considered here, q is greater by 1 than the number q_0, for which the ground-state entropy of the corresponding AF Potts model is zero and the system exhibits an ideal AF long-range ordering at $T=0$, splitting into q_0 sublattices. As for q large the AF Potts model is obviously disordered (paramagnetic, P) already at $T=0$, the problem of whether long-range order exists for (q_0+1)-states models at $T=0$ is a first one to answer in the studies of AF Potts models.

Berker and Kadanoff [3] suggested that AF Potts models can exhibit a low-temperature phase, characterized by an algebraic decay of spin correlations. The square lattice $q=3$ MC simulations [4] suggests, though not unambiguously, that the long-range order may exist. However, MC RG studies [5] demonstrate an absence of a phase transition in this case. Baxter [6] has obtained the analytic result $T_c = 0$ for the square lattice $q=3$ model and Kotecky [7] claimed that it is disordered at $T=0$. MC simulations of the $q=4$ model laid on a triangular lattice [8] suggest that this model is paramagnetic at any T (note that if this is an MC error, it is an error opposite to that of [4] in the sense of its order–disorder favour).

The purpose of this letter is to study the behaviour of the spin correlation function of the two models mentioned above by means of finite-size lattice computations. The results obtained confirm the algebraic decay of spin correlations in both cases [3]. Lattices of size 1–15 (1–10) with free boundary conditions have been used, so apparently the method could err in favour of disordering only. However, trial finite-temperature AF Ising model computations with lattices of the same size have yielded reasonable accuracy. The estimates of the correlation function decrease index obtained correspond to divergent susceptibilities in both cases.

Consider the AF Potts model *correlation function* [1]

$$G(r)=\frac{1}{q-1}\left(q\langle\delta(s_0,s_r)\rangle-1\right). \tag{2}$$

Examining the asymptotics of this function makes it possible to distinguish between the P (paramagnetic), algebraic correlation decay and long-range ordered phases. We consider square lattice calculations in detail. Let us consider square lattices $\mathscr{L}_{sq}(n)$ of size $n\times n$ with free boundary conditions. Define the number $w_t(n)$ of all permissible, i.e. zero-energy, ground-state configurations and the number $w_c(n)$ of those with the values of spins in the two opposite corner sites being equal. The function

$$C(n)=\frac{1}{q-1}\left(q\frac{w_c(n)}{w_t(n)}-1\right) \tag{3}$$

is an estimate for $G(r)$ at $T=0$, and we suggest that in fact it bounds $G(r)$ from below. The algorithm for computing the w_t and w_c is as follows. Let us number sites (i,j) of $\mathscr{L}_{sq}(n)$, defining the function

$$K(i,j)=n(j-1)+i \tag{4}$$

and consider the sequence of sublattices $\mathfrak{T}(k)$ with the boundary $\mathfrak{G}(k)$, $k=1,\ldots,n^2$ (figure 1)

$$\mathfrak{T}(k)=\{(i,j)\colon 1\leq K(i,j)\leq k\} \qquad \mathfrak{T}(n^2)=\mathscr{L}_{sq}(n) \tag{5}$$

$$\mathfrak{G}(k)=\{(i,j)\colon \max(1,k-n)\leq K(i,j)\leq k\}. \tag{6}$$

Let us introduce the designation $\varphi(k)$ for a spin configuration on $\mathfrak{G}(k)$, generated by some ground-state configuration on $\mathfrak{T}(k)$, and the designation $w(n,k;\varphi(k))$ for the number of ground-state configurations on $\mathfrak{T}(k)$, generating the given $\varphi(k)$ on $\mathfrak{G}(k)$. We move one site further towards the upper-right corner B (figure 1), decimating the spin D with the aid of the following relation

$$w(n,k+1;\varphi(k+1))=\sum_{(\varphi(k+1)|\varphi(k))} w(n,k;\varphi(k)) \tag{7}$$

where $(\varphi(k+1)|\varphi(k))$ means compatibility of the configurations $\varphi(k+1)$ and $\varphi(k)$, i.e. their coincidence on the set $\mathfrak{G}(k+1)\cap\mathfrak{G}(k)$. It follows from the compatibility

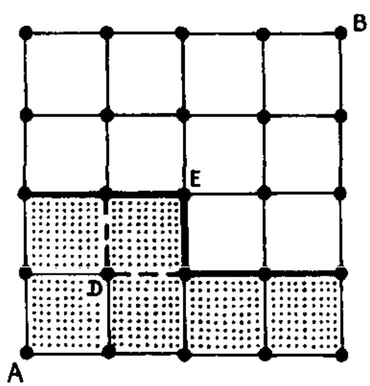

Figure 1. Finite-size square lattice $\mathfrak{L}_{sq}(5)$. Computing the correlation of spins A and B. The sublattice $\mathfrak{T}(13)$ is shown dotted, the boundary $\mathfrak{G}(13)$ is shown by a solid line. On passing from $\mathfrak{T}(12)$, $\mathfrak{G}(12)$ to $\mathfrak{T}(13)$, $\mathfrak{G}(13)$, the site E is added and the spin D is decimated.

condition that the sum in (7) contains not more than $(q-1)$ items and reduces to summing over the spin D (figure 1). Obviously

$$w(n, 1; \varphi(1)) = \delta(s_{(1,1)}, 1) \tag{8}$$

$$w_t(n) = q \sum_{\varphi(n^2)} w(n, n^2; \varphi(n^2)) \tag{9}$$

$$w_c(n) = q \sum_{\varphi(n^2)} \delta(s_{(n,n)}, 1) w(n, n^2; \varphi(n^2)). \tag{10}$$

The formulae (8)-(10) produce an appropriate realization of the well known transfer-matrix method [9], including the enumeration of boundary spin configurations only, for the problem considered. The computations for triangular lattices $\mathscr{L}_{tr}(n)$ are carried out similarly, with $\mathscr{L}_{tr}(n)$ shown in figure 2. The generalization to the $T \geqslant 0$ case is carried out routinely. Computing $C(n)$, (3) requires $\sim(q-1)^n$ bytes of memory ($\sim q^n$ for $T \geqslant 0$) and $\sim n^2(q-1)^n$ CPU time ($\sim n^2 q^n$).

We assume that the asymptotic of $G(r)$ is of the form

$$G(r) \sim \frac{1}{|r|^\eta} \tag{11}$$

where $\eta = \infty$ corresponds to P and $\eta = 0$ to a long-range ordered phase. Similarly,

$$C(n) = \frac{b}{n^{\eta'}} + \ldots \tag{12}$$

and as we have assumed that $C(n)$ bounds $G(r)$ from below, we have $\eta \leqslant \eta'$. Then we obtain a sequence of estimates for the index η'

$$\eta'_n = \frac{[C(n) - C(n-1)][C(n) - C(n+1)]}{C(n)C(n) - C(n-1)C(n+1)}. \tag{13}$$

The values of $C(n)$, η'_n computed for square and triangular lattices are listed in tables

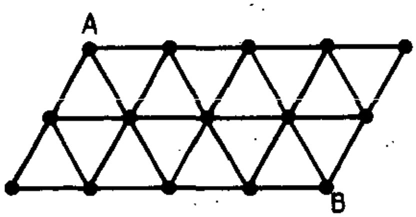

Figure 2. Finite-size triangular lattice $\mathfrak{L}_{tr}(3)$. Computing the correlation of spins A and B.

Table 1. The values of $C(n)$ and η'_n for finite-size square lattices.

n	$C(n)$	η'_n	n	$C(n)$	η'_n
1	1.000 000		9	0.077 504	1.308 32
2	0.500 000	1.777 78	10	0.067 845	1.313 11
3	0.304 878	1.314 71	11	0.060 123	1.316 55
4	0.213 134	1.237 91	12	0.053 825	1.319 13
5	0.161 539	1.249 85	13	0.048 601	1.321 13
6	0.128 779	1.273 06	14	0.044 207	1.322 73
7	0.106 252	1.290 27	15	0.040 466	
8	0.089 884	1.301 25			

Table 2. The values of $C(n)$ and η'_n for finite-size triangular lattices.

n	$C(n)$	η'_n	n	$C(n)$	η'_n
1	1.000 000		6	0.057 640	1.027 83
2	0.333 333	1.844 44	7	0.046 823	1.030 39
3	0.170 108	1.679 14	8	0.039 390	1.039 83
4	0.105 617	1.263 41	9	0.033 965	1.050 92
5	0.074 818	1.079 54	10	0.029 829	

1 and 2, respectively. Hence we obtain the estimates

$$\eta'_{sq} = 1.33 \pm 0.02 \qquad \eta'_{tr} = 1.1 \pm 0.1. \tag{14}$$

Let us note that our results do not rule out the possibility of the existence of a universal index $\eta = \eta_{sq} = \eta_{tr}$.

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References

[1] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
[2] Wu F Y 1984 *J. Appl. Phys.* **55** 2421
[3] Berker A N and Kadanoff L P 1980 *J. Phys. A: Math. Gen.* **13** L259
[4] Grest G S and Banavar J R 1981 *Phys. Rev. Lett.* **46** 1458
[5] Jayaprakash C and Tobochnik J 1982 *Phys. Rev.* B **25** 4890
[6] Baxter R J 1982 *Proc. R. Soc.* A **383** 43
[7] Kotecky R 1985 *Phys. Rev.* B **31** 3088
[8] Grest G S 1981 *J. Phys. A: Math. Gen.* **14** L217
[9] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)